Handout #2

Derivation of the LES demand system

Consumer's problem:

$$\max_{X_i} U = \prod_i (X_i - \gamma_i)^{\beta_i} \text{ s.t. } Y = \sum_i P_i X_i$$

where $\sum_i \beta_i = 1$.

The Lagrange equation:

$$\mathcal{L} = \prod_i (X_i - \gamma_i)^{\beta_i} + \lambda \left(Y - \sum_i P_i X_i \right)$$

First order conditions:

$$\begin{aligned}
\beta_i (X_i - \gamma_i)^{\beta_i - 1} \prod_{j \neq i} \left(X_j - \gamma_j\right)^{\beta_j} &= \lambda P_i \Rightarrow \frac{\beta_i U}{X_i - \gamma_i} = \lambda P_i \\
Y &= \sum_i P_i X_i
\end{aligned}$$

Ratio of two FOC's:

$$\begin{aligned}
\frac{\beta_i \left(X_j - \gamma_j\right)}{\beta_j (X_i - \gamma_i)} &= \frac{P_i}{P_j} \Rightarrow P_j \beta_i X_j - P_j \beta_i \gamma_j = P_i \beta_j (X_i - \gamma_i) \\
P_j X_j &= \frac{P_i \beta_j (X_i - \gamma_i) + P_j \beta_i \gamma_j}{\beta_i}
\end{aligned}$$

Substitute into budget line:

$$\begin{aligned}
Y &= \sum_j P_j X_j = \sum_j \frac{P_i \beta_j (X_i - \gamma_i) + P_j \beta_i \gamma_j}{\beta_i} \\
Y &= \sum_j \frac{\beta_j P_i X_i - \beta_j P_i \gamma_i + \beta_i P_j \gamma_j}{\beta_i} = \frac{\sum_j \beta_j P_i X_i - \sum_j \beta_j P_i \gamma_i + \sum_j \beta_i P_j \gamma_j}{\beta_i} \\
Y &= \frac{P_i X_i \sum_j \beta_j - P_i \gamma_i \sum_j \beta_j + \beta_i \sum_j P_j \gamma_j}{\beta_i} = \frac{P_i X_i - P_i \gamma_i}{\beta_i} + \sum_j P_j \gamma_j
\end{aligned}$$

$$\begin{aligned}
\frac{P_i X_i - P_i \gamma_i}{\beta_i} &= Y - \sum_j P_j \gamma_j \Rightarrow P_i X_i - P_i \gamma_i = \beta_i \left(Y - \sum_j P_j \gamma_j \right) \\
P_i X_i &= P_i \gamma_i + \beta_i \left(Y - \sum_j P_j \gamma_j \right) \Rightarrow X_i = \gamma_i + \frac{\beta_i}{P_i} \left(Y - \sum_j P_j \gamma_j \right)
\end{aligned}$$