Consumer Theories for General Equilibrium Models

Outline

- Utility function and consumer optimization
- Cobb-Douglas utility function
- Optimization with Cobb-Douglas utility function
- Stone-Geary utility function
- Optimization with Stone-Geary utility function
- Nested structure of consumer optimization

Utility function: General

- General form of utility function:

$$U = U\left(X_1, \ldots, X_n\right)$$

where X_i is the quantity of goods consumed; and:

$$\frac{\partial U}{\partial X_i} > 0 \text{ and } \frac{\partial^2 U}{\partial X_i^2} < 0 \text{ for all } i = 1, \ldots, n$$

- For more detail and formal of treatment of the properties of utility function underlying rational consumer's choice, please refer to your microeconomics textbook.

Consumer optimization

- Consumer's problem:

$$\max_{X_1,\ldots,X_n} U\left(X_1,\ldots,X_n\right) \text{ s.t. } \sum_{i=1}^{n} P_i X_i = Y$$

- Write down the Lagrange function:

$$\mathcal{L} = U\left(X_1,\ldots,X_n\right) + \lambda\left(Y - \sum_{i=1}^{n} P_i X_i\right)$$

- First order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial X_i} &= \frac{\partial U}{\partial X_i} - \lambda P_i = 0, \text{ for all } i = 1,\ldots,n \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= Y - \sum_{i=1}^{n} P_i X_i = 0 \end{aligned}$$

Interpretation of first order conditions:

- From ratio of two first order conditions:

$$mrs_{j,i} = \frac{mu_i}{mu_j} = \frac{\frac{\partial U}{\partial X_i}}{\frac{\partial U}{\partial X_j}} = \frac{P_i}{P_j}, \text{ for all } i \neq j$$

  where $mrs_{j,i}$ is the marginal rate of substitution of j for i, and mu_i and mu_j is marginal utility of i and j, respectively.
- When there is only two goods and X_j is the vertical axis and X_i is the horizontal axis:

$$mrs_{j,i} = \text{ slope of indiference curve } = \frac{P_i}{P_j}$$

Consumer optimization with Cobb-Douglas utility function

- Consumer's problem:

$$\max_{X_1,\ldots,X_n} U = \prod_{i=1}^{n} X_i^{\alpha_i} \text{ s.t. } \sum_{i=1}^{n} P_i X_i = Y, \text{ dimana } \sum_i \alpha_i = 1$$

- The Lagrange function:

$$\mathcal{L} = \prod_{i=1}^{n} X_i^{\alpha_i} + \lambda \left(Y - \sum_{i=1}^{n} P_i X_i \right)$$

- First order conditions:

$$\frac{\partial \mathcal{L}}{\partial X_i} = \alpha_i X_i^{\alpha_i - 1} \prod_{j \neq i}^{n} X_j^{\alpha_j} - \lambda P_i = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = Y - \sum_{i=1}^{n} P_i X_i = 0 \quad (2)$$

Consumer optimization with Cobb-Douglas utility function

- Simplifying the first order condition of equation 1:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial X_i} &= \alpha_i X_i^{\alpha_i - 1} \prod_{j \neq i}^{n} X_j^{\alpha_j} - \lambda P_i = 0 \\
&= \alpha_i X_i^{\alpha_i} X_i^{-1} \prod_{j \neq i}^{n} X_j^{\alpha_j} - \lambda P_i = 0 \\
&= \frac{\alpha_i}{X_i} \prod_{i}^{n} X_i^{\alpha_i} - \lambda P_i = 0 \\
&= \frac{\alpha_i}{X_i} U - \lambda P_i = 0, \text{ for } i = 1, \ldots n
\end{aligned}
$$

- Or:

$$\frac{\alpha_i}{X_i} U = \lambda P_i \text{ for } i = 1, \ldots n$$

Consumer optimization with Cobb-Douglas utility function

- Menuliskan rasio dua barang i dan j :

$$\frac{\frac{\alpha_i}{X_i}U}{\frac{\alpha_j}{X_j}U} = \frac{P_i}{P_j} \Rightarrow \frac{\frac{\alpha_i}{X_i}}{\frac{\alpha_j}{X_j}} = \frac{P_i}{P_j} \Rightarrow \frac{\alpha_i}{X_i}\frac{X_j}{\alpha_j} = \frac{P_i}{P_j} \Rightarrow \frac{\alpha_i}{\alpha_j}\frac{X_j}{X_j} = \frac{P_i}{P_j}$$

$$\alpha_i P_j X_j = \alpha_j P_i X_i \Rightarrow P_j X_j = \frac{\alpha_j}{\alpha_i} P_i X_i \Rightarrow P_i X_i = \frac{\alpha_i}{\alpha_j} P_j X_j$$

- Substituting into the budget constraint:

$$\sum_j P_j X_j = Y \Rightarrow \sum_j \frac{\alpha_j}{\alpha_i} P_i X_i = Y \Rightarrow \frac{P_i X_i}{\alpha_i} \sum_j \alpha_j = Y, \ \sum_j \alpha_j = 1$$

$$\frac{P_i X_i}{\alpha_i} = Y$$

$$X_i^* = \alpha_i \frac{Y}{P_i}, \text{ Marshalian demand function for all } i = 1, \ldots n$$

Cobb-Douglas Demand System

$$X_i = \alpha_i \frac{Y}{P_i} \text{ for } i = 1, \ldots, n$$

- Properties:
  - Unitary income elasticity: $\frac{\partial X_i}{\partial Y} \frac{Y}{X_i} = 1$
  - Unitary own-price elasticity: $\frac{\partial X_i}{\partial P_i} \frac{P_i}{X_i} = -1$
  - Zero cross-price elasticity: $\frac{\partial X_i}{\partial P_j} \frac{P_j}{X_i} = 0$
  - Constant budget share: $\frac{P_i X_i}{Y} = \alpha_i$

Stone-Geary utility function

$$U = \prod_i (X_i - \gamma_i)^{\beta_i}$$

- β_i is the marginal budget share and $\sum_i \beta_i = 1$
- γ_i is called the subsistence consumption level.
- $X_i - \gamma_i$ is called supernumary consumption or luxurious consumption, the consumption after subsistence consumption
- Note that when $\gamma_i = 0$ the Stone-Geary utility function is collapsed to Cobb-Douglas

Consumer optimization with Stone-Geary utility function

- Consumer problem:

$$\max_{X_i} U = \prod_i (X_i - \gamma_i)^{\beta_i} \text{ s.t. } Y = \sum_i P_i X_i$$

- The Lagrange equation:

$$\mathcal{L} = \prod_i (X_i - \gamma_i)^{\beta_i} + \lambda \left(Y - \sum_i P_i X_i \right)$$

- First order conditions:

$$\begin{aligned} \beta_i (X_i - \gamma_i)^{\beta_i - 1} \prod_{j \neq i} (X_j - \gamma_j)^{\beta_j} &= \lambda P_i \Rightarrow \frac{\beta_i U}{X_i - \gamma_i} = \lambda P_i \\ Y &= \sum_i P_i X_i \end{aligned}$$

Derivation of the demand function (can be skipped)

- Ratio of two FOC's:

$$\frac{\beta_i\left(X_j-\gamma_j\right)}{\beta_j\left(X_i-\gamma_i\right)} = \frac{P_i}{P_j} \Rightarrow P_j\beta_i X_j - P_j\beta_i\gamma_j = P_i\beta_j\left(X_i-\gamma_i\right)$$

$$P_j X_j = \frac{P_i\beta_j\left(X_i-\gamma_i\right)+P_j\beta_i\gamma_j}{\beta_i}$$

- Substitute into budget line:

$$\begin{aligned} Y &= \sum_j P_j X_j = \sum_j \frac{P_i\beta_j\left(X_i-\gamma_i\right)+P_j\beta_i\gamma_j}{\beta_i} \\ &= \sum_j \frac{\beta_j P_i X_i - \beta_j P_i\gamma_i + \beta_i P_j\gamma_j}{\beta_i} \\ &= \frac{\sum_j \beta_j P_i X_i - \sum_j \beta_j P_i\gamma_i + \sum_j \beta_i P_j\gamma_j}{\beta_i} \\ &= \frac{P_i X_i \sum_j \beta_j - P_i\gamma_i \sum_j \beta_j + \beta_i \sum_j P_j\gamma_j}{\beta_i} = \frac{P_i X_i - P_i\gamma_i}{\beta_i} + \sum_j P_j\gamma_j \end{aligned}$$

Derivation of the demand function (cont)

$$Y = \frac{P_i X_i - P_i \gamma_i}{\beta_i} + \sum_j P_j \gamma_j$$

- Solve for $P_i X_i$:

$$\begin{aligned}
\frac{P_i X_i - P_i \gamma_i}{\beta_i} &= Y - \sum_j P_j \gamma_j \Rightarrow P_i X_i - P_i \gamma_i = \beta_i \left(Y - \sum_j P_j \gamma_j \right) \\
P_i X_i &= P_i \gamma_i + \beta_i \left(Y - \sum_j P_j \gamma_j \right) \text{ in expenditure form} \\
X_i &= \gamma_i + \frac{\beta_i}{P_i} \left(Y - \sum_j P_j \gamma_j \right) \text{ in demand form}
\end{aligned}$$

Some feature of the LES demand function:

$$X_i = \gamma_i + \frac{\beta_i}{P_i}\left(Y - \sum_j P_j \gamma_j\right)$$

- Income elasticity: $\eta_i = \frac{\partial X_i}{\partial Y}\frac{Y}{X_i} = \frac{\beta_i}{P_i}\frac{Y}{X_i} = \frac{\beta_i}{P_i}\frac{Y}{X_i} = \frac{\beta_i}{w_i}$ where $w_i = \frac{P_i X_i}{Y}$ is the budget share.
- Own price elasticity: $\varepsilon_{ii} = \frac{\gamma_i(1-\beta_i)}{x_i} - 1$
- Cross-price elasticity: $\varepsilon_{ij} = -\frac{\beta_i \gamma_j}{x_j}\frac{w_j}{w_i}$
- Frisch parameter: $\Phi = -\frac{Y}{Y - \sum_j P_j \gamma_j}$, is the elasticity of marginal utility of income.
- Queries and discussion:
  - What are the disadvantages of LES demand system? Check the possible value of all the elasticities.
  - How Frisch paramaters varies by income groups (rich and poor) and how it is related to the elasticity of marginal utility of income.

Exercise and group discussion

- Two-goods (X and Y) Stone-Geary Utility function: $U = (X-a)^{\alpha}(Y-b)^{\beta} \Rightarrow (Y-b)^{\beta} = \left(U/(X-a)^{\alpha}\right)$ $\Rightarrow Y = \left(U/(X-a)^{\alpha}\right)^{\frac{1}{\beta}} + b$, defines indifference curve
- Budget constraint: $I = P_X X + P_Y Y \Rightarrow Y = (I - P_x X)/P_y$
- Optimum quantities:
  - $X = a + \frac{\alpha}{P_X}(I - P_X a - P_Y b)$
  - $Y = b + \frac{\beta}{P_Y}(I - P_X a - P_Y b)$
- Note that when $a = 0$ and $b = 0$ the utility is collapsed into a Cobb-Douglas utility function.

Nested structure of consumer's problem

Household problems:

1. Find the optimal combination of sourcing $s \in \{\text{domestic,import}\}$ for any given commodity i, or:

$$\min \sum_s P_{is} X_{is} \text{ subject to } \tilde{X}_i = \left(\sum_s \delta_s X_{is}^{-\rho} \right)^{-\frac{1}{\rho}}$$

2. Find the optimal quantity of goods i to maximize utility subject to budget constraint, or:

$$\max U = \prod_i \tilde{X}_i^{\alpha_i} \text{ subject to } Y = \sum_i \tilde{P}_i \tilde{X}_i$$

Nested structure of consumer's problem (cont.)

1. Solution to: $\min \sum_s P_{is} X_{is}$ subject to $\tilde{X}_i = \left(\sum_s \delta_{si} X_{is}^{-\rho}\right)^{-\frac{1}{\rho}}$:

$$X_{is} = \tilde{X}_i \delta_{is}^{\sigma} \left(\frac{P_{is}}{\tilde{P}_i}\right)^{-\sigma}, \text{ where } \tilde{P}_i = \left(\sum_s \delta_{is}^{\sigma} P_{is}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

2. Solution to: $\max U = \prod_i \tilde{X}_i^{\alpha_i}$ subject to $Y = \sum_i \tilde{P}_i \tilde{X}_i$:

$$\tilde{X}_i = \alpha_i \frac{Y}{\tilde{P}_i}$$